SOME MONOTONICITY RESULTS FOR MINIMIZERS IN THE CALCULUS OF VARIATIONS

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Abstract. We obtain monotonicity properties for minima and stable solutions of general energy functionals of the type

$$\int F(\nabla u, u, x)\, dx$$

under the assumption that a certain integral grows at most quadratically at infinity. As a consequence we obtain several rigidity results of global solutions in low dimensions.

1. Introduction

In this paper we deal with monotonicity properties of minima for quite general energy functionals of the type

$$\int_\Omega F(\nabla u, u, x)\, dx \tag{1.1}$$

where Ω is a domain of $\mathbb{R}^n$. These monotonicity properties are often used for the classification of global minimizers, and therefore play a key role in the regularity theory in the calculus of variations, see for example the case of minimal surfaces theory [22], free boundary problems [2], [9], [16] phase transitions [4] etc. Some applications of our results are given in Section 2.

We consider the case when the domain Ω and the functional F are invariant under translations in a number of directions e_k,..,e_n, and we are interested in monotonicity properties of energy minimizers (or stable solutions) in the class of functions obtained by piecewise Lipschitz domain deformations in these $e_k, .., e_n$ directions. Our main result states that, under rather mild assumptions on F, if there exists a constant $C > 0$ such that for all large R, a minimizer u satisfies

$$\int_{\Omega\cap B_R} |D_p^2 F\,(\nabla u, u, x)|\, |\nabla u|^2\, dx \leqslant CR^2,$$

then u is one-dimensional in each subspace generated by $e_k, .., e_n$ (see Theorem 2.11 for the precise statement).

The general approach to obtain such rigidity results (see for instance the case of minimal surfaces) is to apply the stability inequality (see (2.6)) to a suitable cutoff function. However, this approach becomes often difficult to implement. An example occurs when the functional F becomes singular

near $\partial\Omega$ as in the case of *s-nonlocal minimal surfaces*, $s \in (0,1)$ (see [12]) when the energy functional has the form

$$F(\nabla u, u, x) = |\nabla u|^2 x_1^{1-s}, \quad \Omega = \{x_1 > 0\}.$$

Then the stability inequality does not have a simple form due to the fact that integrations by parts are difficult to handle.

We prove our results inspired by the simple method developed in [25] where we studied global nonlocal minimal surfaces in two dimensions. The main idea is to avoid the precise form of the stability inequality and just compare the energies of u and a translation of itself. In this way we can deal with rather general situations of energy functionals and also consider minimizers directly in the natural class of functions obtained by domain deformations.

We describe briefly the strategy below. We compare the energies of u and

$$\max\{u(x), u(x + te_n)\},$$

and for this we need to modify this comparison function at infinity so that it becomes a compact perturbation of u. The growth condition in the integral above guarantees that the difference between the energy of the perturbed function and the energy of u can be made arbitrarily small. On the other hand, if u is not monotone in the e_n direction then we can modify locally the comparison function above and decrease its energy by a small fixed amount, and this contradicts the minimality of u.

The paper is organized as follows. In Section 2 we state our main theorems and in Section 3 we provide some concrete applications of our results. The main ingredients of the proofs are given in Section 4 where we perform the local analysis, and in Section 5 where we estimate the energy of the perturbations at infinity. The proofs will be completed in Sections 6 and 7. In Section 8 we discuss an explicit 1D example to illustrate better the notion of minimizer and stability in the class of piecewise Lipschitz deformations. Finally in Section 9 we prove several remarks pointed out throughout the paper.

2. Main results

We consider energy functionals as in (1.1) in the case when the domain Ω and the functional F are invariant under translations in the e_n-direction, that is

$$\Omega = \mathscr{U} \times \mathbb{R}, \qquad \mathscr{U} \subseteq \mathbb{R}^{n-1},$$

and F does not depend on the x_n-coordinate.

Points in Ω are denoted by $x = (x', x_n) \in \mathscr{U} \times \mathbb{R}$. We assume that the functional F is convex in with respect to the first variable. Precisely, we suppose that

$$F = F(p, z, x') \in C(\mathbb{R}^n \times \mathbb{R} \times \mathscr{U}), \tag{2.1}$$

and for any $(z, x') \in \mathbb{R} \times \mathscr{U}$, F is C^2 and uniformly convex in p at all p with $p_n \neq 0$.

Furthermore, we assume that $F_{pp} = D_p^2 F$ satisfies the natural growth condition

$$|F_{pp}(p+q, z, x')| \leqslant C\,|F_{pp}(p, z, x')| \tag{2.2}$$

for some $C > 0$, and any p, $q \in \mathbb{R}^n$ with $|q| \leqslant |p_n|/2$.

For any $R > 0$, we introduce the energy functional $\mathscr{E}_R$ defined by

$$\mathscr{E}_R(u) := \int_{\Omega \cap B_R} F(\nabla u(x), u(x), x')\, dx.$$

We study monotonicity properties of suitable minimal or stable solutions for the energy $\mathscr{E}$ among perturbations which are obtained by piecewise domain deformations in the e_n-direction. For this we introduce the following notation:

Definition 2.1. *We say that v is an e_n-Lipschitz deformation of u in B_R if there exists a Lipschitz function ψ with compact support in B_R, and $\|\psi_n\|_{L^\infty(\mathbb{R}^n)} < 1$ such that*

$$v(x) = u(x + \psi(x) e_n) \qquad \forall x \in \Omega.$$

In the notation of Definition 2.1, we have that if u is (locally) Lipschitz then v is (locally) Lipschitz as well.

Definition 2.2. *Let $u \in C^{0,1}(\Omega)$. We say that $v \in C^{0,1}(\Omega)$ is a piecewise e_n-Lipschitz deformation of u in B_R and write*

$$v \in D_R(u)$$

if there exist a finite number $v^{(1)}$,..., $v^{(m)}$ of e_n-Lipschitz deformations of u in B_R such that

$$v(x) = v^{(i)}(x) \qquad \text{for some } i \text{ (depending on } x \in \Omega\text{)}.$$

Also, if all $v^{(i)}$ satisfy

$$v^{(i)}(x) = u(x + \psi^{(i)}(x) e_n) \quad \text{with} \quad \|\psi^{(i)}\|_{C^{0,1}(\Omega)} \leqslant \delta$$

for some $\delta > 0$, we write

$$v \in D_R^\delta(u).$$

We list some elementary properties that follow easily from Definition 2.2:

$$\begin{aligned}
&v, w \in D_R^\delta(u) \quad \Rightarrow \quad \min\{v, w\}, \max\{v, w\} \in D_R^\delta(u); \\
&v \in D_R^\delta(u), \quad w \in D_R^\delta(v) \quad \Rightarrow \quad w \in D_R^{3\delta}(u); \\
&v \in D_R^\delta(u) \quad \Rightarrow \quad \|v - u\|_{L^\infty(\Omega)} \leqslant C\delta \|u\|_{C^{0,1}(\Omega)}; \\
&v \in D_R^\delta(u), \quad u \in C^{1,1}(\Omega) \quad \Rightarrow \quad \|v - u\|_{C^{0,1}(\Omega)} \leqslant C\delta \|u\|_{C^{1,1}(\Omega)}.
\end{aligned} \tag{2.3}$$

Definition 2.3. *We say that $u \in C^{0,1}(\Omega)$ is an e_n-minimizer for $\mathscr{E}$ if for any $R > 0$ we have that $\mathscr{E}_R(u)$ is finite and*

$$\mathscr{E}_R(u) \leqslant \mathscr{E}_R(v), \qquad \forall v \in D_R(u).$$

Remark 2.4. The standard definition in the calculus of variation consists in saying that u is a classical minimizer for $\mathscr{E}$ if it minimizes the energy with respect to compact deformations of the graph of u in the vertical direction (e_{n+1}-direction) that is:

$$\mathscr{E}_R(u) \leqslant \mathscr{E}_R(u+\varphi)$$

for any Lipschitz φ with compact support in $\Omega \cap B_R$. We observe that when $\Omega = \mathbb{R}^n$, e_n-minimality is a weaker condition than classical minimality.

For example any function which is constant in the e_n-direction is always an e_n-minimizer, but not necessarily a classical minimizer.

Our first general monotonicity result is the following:

Theorem 2.5. *Let $u \in C^1(\Omega)$ be an e_n-minimizer for the energy $\mathscr{E}$ with F satisfying* (2.1) *and* (2.2)*.*

If there exists $C > 0$ such that for all large R

$$\int_{\Omega \cap B_R} |F_{pp}(\nabla u, u, x')| \, |\nabla u|^2 \, dx \leqslant CR^2, \tag{2.4}$$

then u is monotone on each line in the e_n-direction, i.e., for any $\bar{x} \in \Omega$, either $u_n(\bar{x} + te_n) \geqslant 0$ or $u_n(\bar{x} + te_n) \leqslant 0$ for any $t \in \mathbb{R}$.

Remark 2.6. If a continuous function u is monotone on each line in $\mathbb{R}^n$ then it is one-dimensional, that is $u = f(x \cdot \xi)$ for some function $f : \mathbb{R} \to \mathbb{R}$ and some unit direction ξ. See Section 9 for a proof.

Our second theorem is a version of Theorem 2.5 for stable critical points of the energy instead of e_n-minimizers. The stability condition we use involves the second variation of $\mathscr{E}$ for deformations of u in the e_n-direction as well as in the vertical e_{n+1}-direction. The precise definition is the following:

Definition 2.7. *We say that w is a piecewise Lipschitz deformation of u in the $\{e_n, e_{n+1}\}$-directions and write*

$$w \in \mathscr{D}^\delta_R(u)$$

if

$$w = v + \varphi \quad \text{with} \quad v \in D^\delta_R(u) \quad \text{and} \quad |\varphi|_{C^{0,1}(\Omega)} \leqslant \delta$$

for some Lipschitz function φ with compact support in $\Omega \cap B_R$.

We remark that here the vertical perturbations φ have compact support in $\Omega \cap B_R$ whereas the e_n-deformations $\psi^{(i)}$ in Definitions 2.1 and 2.2 have compact support in B_R (i.e., if $x \in B_R$ with $x' \in \partial\mathscr{U}$ then $\varphi(x) = 0$ but $\psi^{(i)}(x)$ may be different from 0).

Definition 2.8. *We say that u is a $\{e_n, e_{n+1}\}$-stable solution for $\mathscr{E}$ if for any $R > 0$ and $\epsilon > 0$ there exists $\delta > 0$ depending on R, ϵ and u such that for all $t \in (0, \delta)$ we have that $\mathscr{E}_R(u)$ is finite and*

$$\mathscr{E}_R(w) - \mathscr{E}_R(u) \geqslant -\epsilon t^2, \qquad \forall w \in \mathscr{D}_R^t(u). \tag{2.5}$$

We point out that classical minimality (see Remark 2.4) implies $\{e_n, e_{n+1}\}$-stablity (on the other hand, e_n-minimality and $\{e_n, e_{n+1}\}$-stablity do not imply each other in general). Also, since we allow perturbations in the e_{n+1}-direction in Definition 2.8, then any $\{e_n, e_{n+1}\}$-stable solution is a critical point of the energy functional.

Remark 2.9. In the calculus of variation, it is customary to consider stable solutions of partial differential equations. Classically, a solution (i.e., a critical point of the energy functional) is said to be stable if

$$\liminf_{t \to 0} \frac{\mathscr{E}_R(u + t\varphi) - \mathscr{E}_R(u)}{t^2} \geqslant 0 \tag{2.6}$$

for any Lipschitz function φ supported in B_R.

If $\Omega = \mathbb{R}^n$, $F \in C^2$ and $u \in C^2$, this classical notion of stability is equivalent to the notion of $\{e_n, e_{n+1}\}$-stable solution (for the proof of this, see Section 9).

In the framework given by Definition 2.8 we prove the following result.

Theorem 2.10. *Let $u \in C^{0,1}(\Omega)$ be a $\{e_n, e_{n+1}\}$-stable solution and assume $F \in C^3(\mathbb{R}^2 \times \mathbb{R} \times \mathscr{U})$ satisfies (2.2).*

If the growth condition (2.4) holds then u is monotone in the e_n-direction, i.e. either $u_n \geqslant 0$ or $u_n \leqslant 0$ in Ω.

We observe that the hypotheses in the two theorems above are slightly different and the thesis of Theorem 2.5 is weaker than the one of Theorem 2.10 since in Theorem 2.5 we do not say that $u_n(x)$ has the same sign for all x, but only that, fixed x, $u_n(x + te_n)$ has the same sign for any t.

Our last theorem deals with $\{e_k, ..., e_n\}$-stable solutions, that is, in the definition of stability we allow small piecewise Lipschitz deformations in the $e_k, .., e_n$-directions rather than only the e_n-direction or $\{e_n, e_{n+1}\}$-direction (see Definition 7.3 for a precise statement).

Theorem 2.11. *Assume that*

$$\Omega = \mathscr{U} \times \mathbb{R}^{n-k+1}, \qquad \mathscr{U} \subseteq \mathbb{R}^{k-1},$$

F does not depend on the x_k, ..., x_n coordinates, F satisfies (2.2) and that $F \in C^3$ at all p with $(p_k, ..., p_n) \neq (0, .., 0)$.

If $u \in C^1(\Omega)$ is $\{e_k, ..., e_n\}$-stable and the growth condition (2.4) holds, then u is one-dimensional in any subspace generated by $\{e_k, ..., e_n\}$.

The theorem concludes that for each $(x_1, ..., x_{k-1}) \in \mathscr{U}$, $u(x)$ is one-dimensional (see Remark 2.6) in the remaining variables $(x_k, \ldots, x_n)$. Of course, when $k = n$ the statement becomes trivial.

We point out that the hypothesis above on $\{e_k, ..., e_n\}$-stability for u is in general easily satisfied by critical points of $\mathscr{E}$ which are monotone in the e_n direction, and in fact such critical points are $\{e_k, ..., e_n\}$-minimizers.

We conclude this section with several remarks on the theorems above.

Remark 2.12. The results provided in this paper are in fact even more general: we did not attempt to give the most general conditions possible but rather to emphasize the method of proof (further generalizations will be outlined in subsequent remarks and some of these generalizations turn out to be important in the concrete applications). For instance, we observe that the functional in (1.1) may be generalized to

$$\int_\Omega F(\nabla u, u, x')\,dx + \int_{\partial\Omega} G(u, x')\,d\mathscr{H}^{n-1}, \tag{2.7}$$

where G satisfies the same regularity assumptions as F. The proofs in this case are affected only by minor, obvious modifications.

Remark 2.13. Condition (2.4) may be weakened by allowing logaritmic corrections too. For instance, the right hand side of (2.4) may be replaced by

$$CR^2 \log R$$

or by

$$CR^2(\log R)(\log\log R).$$

More generally, one can define $\ell_0(R) := R$ and recursively

$$\ell_k(R) := \log(\ell_{k-1}(R)) = \underbrace{\log \circ \cdots \circ \log}_{k \text{ times}} R$$

for any $k \in \mathbb{N}$, $k \geqslant 1$. Let also

$$\pi_k(R) := \prod_{j=0}^{k} \ell_j(R). \tag{2.8}$$

Then, instead of (2.4), one may take the weaker condition

$$\int_{\Omega\cap B_R} |F_{pp}(\nabla u, u, x')|\,|\nabla u|^2\,dx \leqslant CR\,\pi_k(R), \tag{2.9}$$

for a given $k \in \mathbb{N}$. For the proof of this fact, see Section 9 (notice that (2.9) boils down to (2.4) if $k = 0$). An energy growth with a logaritmic correction of the type $CR^2 \log R$ was also considered in [23] in the case of semilinear equations.

Remark 2.14. At first glance, Definition 2.2 may look unnecessarily complicated, since one may think that Definition 2.1 suffices for Theorem 2.5. That is, one may think that if u minimizes the energy with respect to any e_n-Lipschitz deformation and (2.4) is satisfied, then u must possess some kind of monotonicity. However this is not the case, as we show by an example in Section 9.

3. Applications

Below we present some direct applications of our results and obtain several rigidity results of global solutions in low dimensions. We remark however that our theorems do not give in general the optimal dimension for these rigidity results.

3.1. **De Giorgi's conjecture.**

As a first application, we obtain a classical one-dimensional symmetry property related to a conjecture of De Giorgi (see [15]):

Theorem 3.1 ([20, 6, 4, 1]). *Let $f \in C(\mathbb{R})$ and $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a solution of $-\Delta u + f(u) = 0$ in the whole of $\mathbb{R}^n$.*

Suppose that:

- *either $n = 2$ and u is stable (according to the notation recalled in Remark 2.9),*
- *or $n = 3$ and $u_3 > 0$.*

Then u is one-dimensional.

Proof. We let $\tilde{F}$ be a primitive of f and we define

$$F(p, z, x) := \frac{1}{2}|p|^2 + \tilde{F}(z).$$

Then, F is clearly convex in p and it satisfies (2.2). It also satisfies (2.4): when $n = 2$ this simply follows from the fact that $|B_R| \leqslant CR^2$, and when $n = 3$ it is a consequence of Theorem 5.2 in [1].

Now we apply Theorem 2.10 and obtain that u is one-dimensional. □

We stress that the proof of Theorem 3.1 that we give here is based on domain perturbations and it does not use some of the basic ingredients exploited in the existing literature: e.g., differently from [6, 4, 1], it does not use any Liouville type result, differently from [20] it does not use the Ekeland's variational principle, differently from [18] it makes no use of any complex structure, differently from [24] no costruction of barriers is needed, and differently from [17, 19] no geometric Poincaré inequality is exploited.

Remark 3.2. The one-dimensional results related to the Conjecture of De Giorgi in dimensions 2 and 3 may be extended to a very broad class of operators and nonlinearities: see Theorems 1.1 and 1.2 in [19]. We remark that our Theorem 2.10 also implies Theorems 1.1 and 1.2 in [19] (at least in case of smooth nonlinearities; for a proof of this fact see Section 9).

3.2. **Fractional De Giorgi conjecture.**

The one-dimensional symmetry of Theorem 3.1 has a counterpart in the fractional Laplace framework, that may be also obtained as a consequence of the results of this paper:

Theorem 3.3 ([10, 11, 7, 8]). *Let $s \in (0, 1)$, $f \in C(\mathbb{R})$ and $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a solution of $(-\Delta)^s u + f(u) = 0$ in the whole of $\mathbb{R}^n$.*

Suppose that:

- *either* $n = 2$ *and* u *is stable (according to the notation recalled in Remark 2.9),*
- *or* $n = 3$*,* $s \in [1/2, 1)$ *and* $u_3 > 0$*.*

Then u *is one-dimensional.*

Proof. We use the extension result in [13] and therefore we reduce this problem to an energy functional in $(0, +\infty) \times \mathbb{R}^n$ as the one in (2.7) with

$$\mathscr{U} := (0, +\infty) \times \mathbb{R}^{n-1}, \quad F(p, z, x) := x_1^{1-2s}|p|^2, \qquad G(z, x) := \tilde{F}(z),$$

where $\tilde{F}$ is a primitive of f (notice that n in Theorem 2.10 must be replaced by $n + 1$ for this application). Then, the desired energy growth follows from [7, 8], according to which

$$\mathscr{E}_R(u) \leqslant \begin{cases} CR^{n-\min\{2s,1\}} & \text{if } s \neq 1/2, \\ CR^{n-1} \log R & \text{if } s = 1/2. \end{cases}$$

Therefore, (2.4) is satisfied when $n = 2$ and also when $n = 3$ and $s \in (1/2, 1)$ (on the other hand, when $n = 3$ and $s = 1/2$, (2.9) is satisfied and one has to make use of Remark 2.13). Thus we obtain Theorem 3.3 as a consequence of Theorem 2.10 and Remark 2.6. □

3.3. **Minimal surfaces.** Minimal surfaces in $\mathbb{R}^n$ can be thought as boundaries of sets $E \subset \mathbb{R}^n$ that minimize the BV-norm or the perimiter (see [22])

$$Per(\chi_E) := \int |D\chi_E|\, dx.$$

Although the functional F does not satisfy precisely the conditions of our theorems, the methods of proof of the next two sections easily apply to this case as well. Then condition (2.4) reads

$$Per(\chi_E, B_R) = \int_{B_R} |D\chi_E|\, dx \leqslant CR^2.$$

On the other hand the perimeter of a minimal surface in B_R is bounded by the surface area of ∂B_R, that is CR^{n-1}, hence the only global minimal surfaces in $\mathbb{R}^3$ are the hyperplanes (one-dimensional).

3.4. **Nonlocal minimal surfaces.** As mentioned in the Introduction we discuss a result on nonlocal perimeters which was the original motivation for the techniques developed in this paper, see [25].

Given a bounded domain $\Omega \subset \mathbb{R}^n$, the minimization of the following functional was introduced in [12]:

$$\mathrm{Per}_s(E, \Omega) := L(E \cap \Omega, \mathbb{R}^n \setminus E) + L(E \setminus \Omega, \Omega \setminus E),$$

where $s \in (0, 1)$ and for any disjoint measurable sets A and B,

$$L(A, B) := \int_A \int_B \frac{dx\, dy}{|x - y|^{n+s}}.$$

The regularity of s-minimal surfaces (i.e. of the boundary of a set E which minimizes $\mathrm{Per}_s(\cdot,\Omega)$ among all the measurable sets that agree with E outside Ω) and of s-minimal cones (i.e. of s-minimal surfaces E such that are invariant under dilations) has been studied in some recent papers, such as [12, 14, 25, 5]. In particular, a complete regularity theory holds in the plane, according to the following result, that may also be obtained as a byproduct of the results in this paper:

Theorem 3.4 ([25])**.** *If E is an s-minimal cone in $\mathbb{R}^2$, then E is a half-plane.*

Proof. By the extension result in Section 7 of [12], we reduce the problem to a variational energy in $(0,+\infty)\times\mathbb{R}^2$, with

$$F(p,z,x) := x_1^{1-s}|p|^2,$$

for a minimizer homogenous of degree 0. Then, (2.4) easily follows in dimension $n = 2$, and so we may use again Theorem 2.10. □

3.5. **Two-phase free boundary problem.** This classical free boundary problem (see [4], [3]) consists in minimizing the energy

$$\int |\nabla u|^2 dx + |\{u > 0\}|.$$

In this case condition (2.4) becomes

$$\int_{B_R} |\nabla u|^2 dx \leqslant CR^2,$$

which is clearly satisfied by a Lipschitz minimizer in dimension $n = 2$. In conclusion, in $\mathbb{R}^2$ any Lipschitz minimizer for the two-phase problem must be one-dimensional.

3.6. **Thin one-phase problem.** In this free boundary problem we minimize the following energy in $\mathbb{R}^{n+1}_+$ (see [16])

$$\int_{\mathbb{R}^{n+1}_+} |\nabla u|^2 dX + \mathscr{H}^n(\{u(x,0) > 0\}),$$

where we denote the points in $\mathbb{R}^{n+1}$ by $X = (x, x_{n+1})$.

Our results imply that in dimension $n = 2$, any homogenous minimizer must be one-dimensional in the x variable. This follows easily from (2.4) since, due to the scaling of the energy, any homogeneous minimizer must be homogenous of degree $1/2$.

4. Local perturbations

In this section we show that in general we can perturb locally

$$\max\{u(x), u(x + te_n)\}$$

into a function with lower energy.

The first lemma states that the maximum of two functions that form an angle at an intersection point cannot be an e_n-minimizer for $\mathscr{E}$ (this fact uses the strict convexity of F in the p variable).

Lemma 4.1. *Assume $0 \in \Omega$ and u, v are C^1-functions such that*

$$u(0) = v(0) \text{ and } v_n(0) < 0 < u_n(0). \tag{4.1}$$

Then $g := \max\{u, v\}$ is not an e_n-minimizer for $\mathscr{E}$ in any ball B_η.

Remark 4.2. In our setting, the transversal intersection described analytically by (4.1) can be obtained whenever u is not monotone on each line along the e_n-direction. In this case we may reduce to the case in which $u(\bar{x} + a_1 e_n) < u(\bar{x} + a_2 e_n)$ and $u(\bar{x} + a_2 e_n) > u(\bar{x} + a_3 e_n)$, with $a_1 < a_2 < a_3$. Let $c_i := u(\bar{x} + a_i e_n)$. Then, by Sard's theorem we can find a regular value $c \in \big(\max\{c_1, c_3\},\, c_2\big)$ of u, thus we may find $\alpha_c \in (a_1, a_2)$ and $\beta_c \in (a_2, a_3)$ such that $u(\bar{x}+\alpha_c e_n) = c = u(\bar{x}+\beta_c e_n)$ and $u_n(\bar{x}+\alpha_c e_n) > 0 > u_n(\bar{x} + \beta_c e_n)$. Then, the setting of (4.1) is fulfilled by supposing, up to translations, that $\bar{x} + \alpha_c e_n = 0$ and by taking $v(x) := u(x + (\beta_c - \alpha_c)e_n)$.

Proof of Lemma 4.1. Assume by contradiction that g is an e_n-minimizer in some small ball B_η. We define $F_0(p) := F(p, 0, 0)$, and we claim that we may reduce to the case in which

$$F_0(\nabla u(0)) = F_0(\nabla v(0)).$$

To see this we notice that the property of minimality is not affected after subtracting a linear functional from F. Precisely if

$$\tilde{F}(p, z, x) := F(p, z, x) - p_0 \cdot p,$$

and $\tilde{\mathscr{E}}_R$ is the associated energy functional for $\tilde{F}$ in B_R then

$$\mathscr{E}_R(f) - \tilde{\mathscr{E}}_R(f) = \int_{B_R} p_0 \cdot \nabla f \, dx = \int_{\partial B_R} f \, p_0 \cdot \nu.$$

That is, $\tilde{\mathscr{E}}_R(f)$ and $\mathscr{E}_R(f)$ only differ by a term depending on the boundary values of f. Consequently, if f is an e_n-minimizer for $\mathscr{E}$, it is also an e_n-minimizer for $\tilde{\mathscr{E}}$.

Also, by possibly translating F in the z-variable, we may assume that $u(0) = v(0) = 0$. Now, for small $r > 0$, we consider the rescalings

$$u_r(x) := r^{-1} u(rx), \quad v_r(x) := r^{-1} v(rx)$$

and we define $g_r(x) := \max\{u_r(x), v_r(x)\}$. Then, g_r is an e_n-minimizer for the rescaled functional

$$F_r(p, z, x) := F(p, rz, rx)$$

in $B_{\eta/r}$. As $r \to 0^+$ then the following limits hold uniformly on compact sets:

$$\begin{aligned} &F_r \to F_0(p), \\ &u_r(x) \to u_0(x) := \nabla u(0) \cdot x, \qquad \nabla u_r \to \nabla u_0, \\ &v_r(x) \to v_0(x) := \nabla v(0) \cdot x, \qquad \nabla v_r \to \nabla v_0. \end{aligned} \tag{4.2}$$

So we let

$$g_0 = \max\{u_0, v_0\}.$$

From the strict convexity of F in the p variable we see that g_0 is not a minimizer for F_0. Indeed we first construct h_0,

$$h_0 := 1 + \alpha u_0 + (1-\alpha)v_0 - \rho_R(x'), \qquad \rho_R(x') := \max\{0, |x'| - R\}$$

for some $\alpha \in (0,1)$ small and R large. Then

$$\max\{g_0, h_0\},$$

coincides with g_0 outside B_{R+C} and notice that in B_R we are cutting the graphs of two transversal linear functions by a single one. This function has lower energy for F_0 than the one of g_0 provided that we choose R sufficiently large.

By using the uniform convergence in (4.2), we see that

$$h_r := \max\{g_r, h_0\},$$

has lower energy for F_r than the one of g_r.

Scaling back, we have that $h_\star(x) := rh_r(x/r)$ has less energy for F in $B_{r(R+C)} \subseteq B_\eta$ than the one of g. To reach a contradiction, it remains to check that $h_\star$ is indeed an allowed perturbation according to Definition 2.2. This is equivalent to say that h_r is a piecewise Lipschitz domain deformation of g_r with the Lipschitz norm bounded by δ.

To obtain this, we use our hypothesis $\nabla u_0 \cdot e_n > 0 > \nabla v_0 \cdot e_n$ and the uniform convergence (in C^1) of u_r and v_r to u_0 respectively v_0. Then, by the Implicit Function Theorem, the part of the graph of h_r where $h_0 > g_r$ is obtained from u_r by a Lipschitz domain deformation with Lipschitz norm less than δ, provided that α is chosen sufficiently small. $\square$

Remark 4.3. In the proof we also showed that if u, v are C^1 functions with $u(0) = v(0)$ and $\nabla u(0) \neq \nabla v(0)$ then $g := \max\{u, v\}$ is not a classical minimizer for $\mathscr{E}$ in B_η.

The second lemma deals with perturbations for $\max\{u(x), u(x+te_n)\}$ (for small t) near a non-degenerate point on $\{u_n = 0\}$.

Lemma 4.4. *Assume that $u \in C^2(\Omega)$ is a critical point for the energy $\mathscr{E}$ in a neighborhood of the origin and the functional $F \in C^2$ in a neighborhood of $(\nabla u(0), u(0), 0)$. Assume that*

$$u_n(0) = 0, \quad \nabla u_n(0) \neq 0 \tag{4.3}$$

and let

$$w(x) := \max\{u(x), u(x + te_n)\}. \tag{4.4}$$

Then, for any $\eta > 0$, there exists a Lipschitz function φ with compact support in B_η such that

$$\mathscr{E}_\eta(w + t\varphi) - \mathscr{E}_\eta(w) \leqslant -ct^2 \qquad \text{for all } t \text{ small,}$$

for some small $c > 0$ depending on u, F and η.

Proof. Let

$$v(x) := \frac{u(x+te_n)-u(x)}{t} \tag{4.5}$$

and notice that

$$\|v-u_n\|_{C^{0,1}(B_\eta)} = o(1) \quad \text{as } t\to 0. \tag{4.6}$$

Given a Lipschitz function g we use that $F\in C^2$ in the (p,z) variables and obtain

$$\mathscr{E}_\eta(u+tg) = \mathscr{E}_\eta(u) + tL(g) + t^2 Q(g) + o(t^2)$$

with

$$L(g) := \int_{B_\eta} F_p\cdot\nabla g + F_z\, g\, dx,$$

$$Q(g) := \int_{B_\eta} G(\nabla g, g, x)\, dx = \int_{B_\eta} (\nabla g)^T F_{pp}\nabla g + 2gF_{pz}\cdot\nabla g + F_{zz}g^2\ dx.$$

In the integrals above the function F and its derivatives are evaluated at $(\nabla u, u, x)$ and the constant in the error term $o(t^2)$ depends on u, F and $\|g\|_{C^{0,1}(B_\eta)}$. Since u is a critical point for $\mathscr{E}$ we see that if φ has compact support in B_η then

$$\mathscr{E}_\eta(u+tv^+ + t\varphi) - \mathscr{E}_\eta(u+tv^+) = t^2(Q(v^+ +\varphi) - Q(v^+)) + o(t^2). \tag{4.7}$$

From (4.4) and (4.5), we see that

$$w = u + tv^+. \tag{4.8}$$

Also, we claim that, if η is sufficiently small,

$$Q(v^+) - Q(u_n^+) = o(1) \ \text{ and } \ Q(v^+ +\varphi) - Q(u_n^+ +\varphi) = o(1). \tag{4.9}$$

We prove the first relation, the second being analogous. For this, we fix $\mu>0$ and we define $\mathscr{A}_\mu := B_\eta\cap\{|u_n|\leqslant\mu\}$ and $\mathscr{B}_\mu := B_\eta\cap\{|u_n|>\mu\}$. From (4.6), we have that

$$\lim_{t\to0}\|v^+ - u_n^+\|_{C^{0,1}(\mathscr{B}_\mu)} = \lim_{t\to0}\|v-u_n\|_{C^{0,1}(\mathscr{B}_\mu)} = 0. \tag{4.10}$$

On the other hand, since $\nabla u_n(0)\neq0$, for small η we have that the measure of $\mathscr{A}_\mu$ is (at most) of the order of μ. This and (4.6) yield that

$$\lim_{t\to0}|Q(v^+) - Q(u_n^+)| \leqslant C\mu$$

and so (4.9) follows since μ can be taken arbitrarily small.

From (4.9) we see that if η is sufficiently small we can replace v^+ by u_n^+ in the right hand side of (4.7): accordingly, recalling also (4.8), we obtain

$$\mathscr{E}_\eta(w+t\varphi) - \mathscr{E}_\eta(w) = t^2(Q(u_n^+ +\varphi) - Q(u_n^+)) + o(t^2). \tag{4.11}$$

On the other hand u_n, 0 and G satisfy the hypotheses of Remark 4.3, hence u_n^+ is not a minimizer of Q. Thus we can choose φ such that

$$Q(u_n^+ +\varphi) \leqslant Q(u_n^+) - c$$

for some small $c > 0$, possibly depending on u, F and η. So, by (4.11),

$$\mathscr{E}_\eta(u + tv^+ + t\varphi) - \mathscr{E}_\eta(u + tv^+) \leqslant -\frac{c}{2}t^2$$

for all small t. □

Remark 4.5. If $\nabla u(0) \neq 0$ then the function $w + t\varphi$ can be interpreted (via the Implicit Function Theorem) as a Lipschitz domain deformation of w in the $\nabla u(0)$-direction (see Definition 2.1) and the $C^{0,1}$-norm of the deformation is bounded by Ct. Notice that, in general, the $\nabla u(0)$-direction and the e_n-direction are different.

The non-degeneracy hypothesis $\nabla u_n \neq 0$ of Lemma 4.4 can be checked easily from Hopf lemma if $F \in C^3$ in a neighborhood of $\nabla u(0)$, as next result points out.

Lemma 4.6. *Assume that $u \in C^1(\Omega)$ is a critical point for $\mathscr{E}$ and $F \in C^3$ in a neighborhood of $(\nabla u(0), u(0), 0)$. If $u_n(0) = 0$ and u_n does not vanish identically in a neighborhood of 0 then there exists a point x_0 close to 0 such that $u_n(x_0) = 0$, $\nabla u_n(x_0) \neq 0$.*

Proof. Since u is a critical function for $\mathscr{E}$ then it satisfies the elliptic equation

$$G(D^2u, Du, u, x') := \mathit{div}\, F_p(\nabla u, u, x') - F_z(\nabla u, u, x') = 0.$$

From the De Giorgi-Nash-Moser theorem and the Schauder estimates (see [21]) it follows that if u is locally Lipschitz and $F \in C^{2,\alpha}$ then $u \in C^{2,\alpha}$ and the equation above is satisfied there in the classical sense. If $F \in C^3$ then $G \in C^1$ hence by differentiating the equation in the e_n-direction we see that $v = u_n$ satisfies the linearized equation (in the viscosity sense)

$$Lv := G_{ij}v_{ij} + G_{p_i}v_i + G_z v = 0,$$

where the derivatives of G are evaluated at (D^2u, Du, u, x'). Since v does not vanish identically we can apply Hopf lemma to v at a point $x_0 \in \{v = 0\}$ which admits a tangent ball from either $\{v > 0\}$ or $\{v < 0\}$. □

5. Perturbations at infinity

For all R large we define the Lipschitz continuous function ψ_R with compact support in $\mathbb{R}$ given by

$$\psi_R(s) := \begin{cases} 1, & 0 \leqslant s \leqslant \sqrt{R}, \\ 2 - \dfrac{2\log s}{\log R}, & \sqrt{R} < s \leqslant R, \\ 0, & s > R. \end{cases} \tag{5.1}$$

Notice that

$$\psi'_R(s) = \begin{cases} 0, & s \in (0, \sqrt{R}) \cup (R, \infty), \\ \\ \dfrac{-2}{s \log R}, & s \in (\sqrt{R}, R). \end{cases} \tag{5.2}$$

For $0 < t \leqslant \sqrt{R}/4$, we define a bi-Lipschitz change of coordinates:

$$x \mapsto y(x) := x + t\psi_R(|x|)e_n$$

and let

$$u^+_{R,t}(y) = u(x).$$

Notice that $u^+_{R,t}(x)$ coincides with $u(x - te_n)$ in $B_{\sqrt{R}/2}$ and with $u(x)$ outside B_R. Next we estimate $\mathscr{E}_R(u^+_R)$ in terms of $\mathscr{E}_R(u)$. We have

$$D_x y = I + A,$$

with

$$A(x) = t\,\psi'_R(|x|) \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{x_1}{|x|} & \frac{x_2}{|X|} & \cdots & \frac{x_n}{|x|} \end{pmatrix}$$

and

$$\|A\| \leqslant t|\psi'_R(|x|)| \ll 1.$$

Notice that

$$D_y x = (I + A)^{-1} = I - \frac{1}{1 + trA} A.$$

We have,

$$\nabla_y u^+_R = \nabla_x u \; D_y x, \quad dy = (1 + trA)dx,$$

thus

$$\begin{aligned} &\int_{\Omega \cap B_R} F(\nabla_y u^+_{R,t}, u^+_{R,t}, y')dy \\ &\quad = \int_{\Omega \cap B_R} F\left(\nabla_x u \; \left(I - \frac{1}{1 + trA} A\right), u, x'\right)(1 + trA)dx. \end{aligned}$$

We bound the right hand side from above by using that $|(pA)| \leqslant |p \cdot e_n|/4$ which together with hypothesis (2.2) for F gives that

$$F\left(p \; \left(I - \frac{1}{1 + trA} A\right), z, x'\right)(1 + trA)$$

is bounded above by

$$F(p, z, x')(1 + trA) - F_p(p, z, t) \cdot (pA) + C|F_{pp}(p, z, t)||pA|^2. \tag{5.3}$$

By writing the same inequality for $u^-_{R,t}$ which is defined as $u^+_{R,t}$ with t replaced by $-t$, thus A is replaced by $-A$ in the formulas above, we obtain

$$\begin{aligned} & \mathscr{E}_R(u^+_{R,t}) + \mathscr{E}_R(u^-_{R,t}) - 2\mathscr{E}_R(u) \\ & \leqslant C \int_{\Omega\cap B_R} |F_{pp}(\nabla u, u, x')||\nabla u|^2|A|^2 dx \\ & \leqslant C\frac{t^2}{(\log R)^2}\int_{\Omega\cap\left(B_R\setminus B_{\sqrt{R}}\right)} \frac{|F_{pp}(\nabla u, u, x')||\nabla u|^2}{|x|^2}dx. \end{aligned} \tag{5.4}$$

We denote by

$$a(r) := \int_{\Omega\cap B_r} |F_{pp}(\nabla u, u, x')||\nabla u|^2 dx \tag{5.5}$$

and by hypothesis (2.4) we know that $a(r) \leqslant Cr^2$. Then the last integral in (5.4) is controlled, in polar coordinates, by

$$\int_{\sqrt{R}}^{R} a'(r)r^{-2}dr \leqslant a(R)R^{-2} + 2\int_{\sqrt{R}}^{R} a(r)r^{-3} \leqslant C\log R. \tag{5.6}$$

From (5.4) and (5.6) we conclude that

$$\limsup_{R\to+\infty}\sup_{t\in(0,\sqrt{R}/4)} t^{-2}\Big(\mathscr{E}_R(u^+_{R,t}) + \mathscr{E}_R(u^-_{R,t}) - 2\mathscr{E}_R(u)\Big) \leqslant 0. \tag{5.7}$$

6. Proofs of Theorems 2.5 and 2.10

Proof of Theorem 2.5. Since u is an e_n-minimizer we know that

$$\mathscr{E}_R(u^+_{R,t}) \geqslant \mathscr{E}_R(u).$$

This and (5.7) imply that, for any fixed t, we have

$$\lim_{R\to+\infty} \mathscr{E}_R(u^-_{R,t}) - \mathscr{E}_R(u) = 0. \tag{6.1}$$

Now we recall the integral formula

$$\mathscr{E}_R(\max\{u^-_{R,t}, u\}) + \mathscr{E}_R(\min\{u^-_{R,t}, u\}) = \mathscr{E}_R(u^-_{R,t}) + \mathscr{E}_R(u), \tag{6.2}$$

and we make use of the minimality of u, which implies that

$$\mathscr{E}_R(\min\{u^-_{R,t}, u\}) \geqslant \mathscr{E}_R(u). \tag{6.3}$$

By (6.1), (6.2) and (6.3) we find

$$\lim_{R\to+\infty} \mathscr{E}_R(v_{R,t}) - \mathscr{E}_R(u) = 0, \tag{6.4}$$

with

$$v_{R,t} := \max\{u^-_{R,t}, u\}. \tag{6.5}$$

Notice that

$$v_{R,t} = \max\{u(x), u(x+te_n)\} \quad \text{in } B_{\sqrt{R}/4},$$

and $v_{R,t} \in D^t_R(u)$.

Now assume by contradiction that $u \in C^1(\Omega)$ is not monotone on a line in the e_n-direction. Then we can find $t > 0$ so that $u(x)$, $u(x + te_n)$ satisfy the hypotheses of Lemma 4.1 (say, at some point $x_0 \in \Omega$, recall Remark 4.2).

Thus we can perturb $v_{R,t}$ locally near x_0 into $\tilde{v}_{R,t}$ such that

$$\mathscr{E}_R(\tilde{v}_{R,t}) \leqslant \mathscr{E}_R(v_{R,t}) - c \tag{6.6}$$

for some fixed $c > 0$ depending only on u. From (6.4) and (6.6) we contradict the minimality of u as $R \to +\infty$. □

Proof of Theorem 2.10. We argue as above and use Lemma 4.4 instead. Given $\epsilon > 0$ we choose R large such that

$$\mathscr{E}_R(u^+_{R,t}) + \mathscr{E}_R(u^-_{R,t}) - 2\mathscr{E}_R(u) \leqslant \epsilon t^2.$$

Since u is $\{e_n, e_{n+1}\}$-stable we have

$$\mathscr{E}_R(w) \geqslant \mathscr{E}_R(u) - \epsilon t^2 \quad \forall w \in \mathscr{D}^t_R(u),$$

for all t small enough (the first relation above comes from (5.7) and the second one from Definition 2.8). Then, using also (6.2) and (6.5), we obtain

$$\mathscr{E}_R(v_{R,t}) - \mathscr{E}_R(u) \leqslant 3\epsilon t^2.$$

If u_n changes sign in Ω then from Lemma 4.6 we can find a point $x_0 \in \Omega$ such that u satisfies the hypothesis of Lemma 4.4 at x_0. Thus we can perturb $v_{R,t}$ locally near x_0 into $\tilde{v}_{R,t}$ such that

$$\mathscr{E}_R(\tilde{v}_{R,t}) \leqslant \mathscr{E}_R(v_{R,t}) - ct^2, \qquad \tilde{v}_{R,t} \in \mathscr{D}^{Ct}_R(u),$$

for some $c, C > 0$ depending only on u. In conclusion

$$\mathscr{E}_R(\tilde{v}_{R,t}) \leqslant \mathscr{E}_R(u) + (3\epsilon - c)t^2,$$

and we contradict the stability inequality if we choose $\epsilon \ll c$. □

7. Proof of Theorem 2.11

In this section we assume that the domain Ω and the functional F are invariant under translations in the e_k,..., e_n-directions.

We define the notion of u to be stable with respect to piecewise Lipschitz deformations in all directions generated by $\{e_k, ..., e_n\}$ (but not with respect to vertical e_{n+1} deformations as in Definition 2.7). Below to give a precise definition of $\{e_k, ..., e_n\}$-stability, we modify Definitions 2.1 and 2.2 according to the following notation:

Definition 7.1. *We say that v is an $\{e_k, ..., e_n\}$-Lipschitz deformation of u in B_R if there exist Lipschitz functions $\psi^{(k)}, ..., \psi^{(n)}$ with compact support in B_R, and*

$$\sum_{k \leqslant i,j \leqslant n} \|\psi^{(i)}_j\|^2_{L^\infty(\mathbb{R}^n)} < 1 \tag{7.1}$$

such that

$$v(x) = u(x + \psi^{(k)}(x)e_k + ... + \psi^{(n)}(x)e_n).$$

We remark that, under condition (7.1), the map

$$x \mapsto x + \psi^{(k)}(x)e_k + \dots + \psi^{(n)}(x)e_n$$

is a diffeomorphism.

Definition 7.2. *Let $u \in C^{0,1}(\Omega)$. We say that $v \in C^{0,1}(\Omega)$ is a piecewise $\{e_k, \dots, e_n\}$-Lipschitz deformation of u in B_R and write*

$$v \in D_{R,k}(u)$$

if there exist a finite number $v^{(1)}, \dots, v^{(m)}$ of $\{e_k, \dots, e_n\}$-Lipschitz deformations of u in B_R such that

$$v(x) = v^{(i)}(x) \qquad \text{for some } i \text{ (depending on } x\text{)}.$$

Also, if all $v^{(i)}$ satisfy

$$v^{(i)}(x) = u(x + \psi^{(i,k)}(x)e_k + \dots + \psi^{(i,n)}(x)e_n) \quad \text{with} \quad \|\psi^{(i,j)}\|_{C^{0,1}(\Omega)} \leqslant \delta$$

for some $\delta > 0$, we write

$$v \in D^{\delta}_{R,k}(u).$$

Definition 7.3. *We say that u is $\{e_k, \dots, e_n\}$-stable for $\mathscr{E}$ if for any $R > 0$ and $\epsilon > 0$ there exists $\delta > 0$ depending on R, ϵ and u such that for all $t \in (0, \delta)$ we have that $\mathscr{E}_R(u)$ is finite and*

$$\mathscr{E}_R(v) - \mathscr{E}_R(u) \geqslant -\epsilon t^2, \qquad \forall v \in D^t_{R,k}(u).$$

Notice that Definitions 2.8 and 7.3 are quite different, since vertical perturbations are allowed in Definition 2.8 but not in Definition 7.3. On the other hand, Definition 7.3 allows for horizontal perturbations in $(n-k+1)$-horizontal directions, while only one horizontal direction may be perturbed in Definition 2.8.

Remark 7.4. We point out that if $u \in C^1(\Omega)$ is $\{e_k, \dots, e_n\}$-stable and u_k,...,u_n do not vanish all at some point then u is a critical point for $\mathscr{E}$ in a neighborhood of that point (because any vertical perturbation $u + \epsilon\psi$ may be written in this case as a horizontal perturbation in the span of $\{e_k, \dots, e_n\}$, due to the Implicit Function Theorem).

Proof of Theorem 2.11. The proof of Theorem 2.11 follows as before from Lemma 4.4, Remark 4.5 and Lemma 4.6. First we may suppose that $k < n$, otherwise the statement is trivial.

Let Y_0 be a point in $\mathscr{U} \subset \mathbb{R}^{k-1}$ and we want to show that $\tilde{u}$ is one-dimensional where

$$\tilde{u}(x_k, .., x_n) := u(Y_0, x_k, .., x_n).$$

Assume that $0 \in \mathbb{R}^{n-k+1}$ is such that $\nabla \tilde{u}(0)$ is nonzero and it points in the e_k direction. Then we may apply Lemma 4.4 and Remark 4.5 in the e_{k+1},.., e_n directions and conclude that $\tilde{u}$ is constant in a neighborhood of 0 in all these directions. Then the set

$$\big\{(x_{k+1}, .., x_n) \text{ s.t. } \tilde{u}(0, x_{k+1}, .., x_n) = \tilde{u}(0), \quad \nabla\tilde{u}(0, x_{k+1}, .., x_n) = \nabla\tilde{u}(0)\big\}$$

is both open and closed, hence the level set $\{\tilde u = \tilde u(0)\}$ contains the hyperplane $0 \times \mathbb{R}^{n-k}$. This argument shows that at all points where $\nabla \tilde u$ is nonzero, the gradient must point in the e_k direction, thus $\tilde u$ depends only on the x_k variable. □

Remark 7.5. We point out that condition (2.2) on F can be weakened in Theorems 2.10 and 2.11. Since we only need (5.7) as $t \to 0$ we see from Section 5 that it suffices to have that

$$x \mapsto \sup_{|p-\nabla u|\leqslant |\nabla u|/2} |F_{pp}(p,u,x')|\ |\nabla u|^2$$

is a locally integrable function.

We conclude this section with a version of Theorem 2.5 for e_n-minimiziers with respect to piecewise Lipschitz perturbations with norm bounded by δ.

Definition 7.6. *We say that $u \in C^{0,1}(\Omega)$ is a $\{\delta, e_n\}$-minimizer for $\mathscr{E}$ if for any $R > 0$ we have that $\mathscr{E}_R(u)$ is finite and*

$$\mathscr{E}_R(u) \leqslant \mathscr{E}_R(v), \qquad \forall v \in D^\delta_R(u).$$

Theorem 7.7. *Let $\delta > 0$ and $u \in C^1(\Omega)$ be a $\{\delta, e_n\}$-minimizer for the energy $\mathscr{E}$ with F satisfying* (2.1) *and* (2.2).

If (2.4) *is satisfied, then u is monotone on each segment in the e_n-direction of length less than 2δ, i.e., for any $\bar x \in \Omega$, either $u_n(\bar x + te_n) \geqslant 0$ or $u_n(\bar x + te_n) \leqslant 0$ for any $t \in (-\delta, \delta)$.*

The proof of Theorem 7.7 is identical to the one of Theorem 2.5, we just need to choose $|t| < \delta$.

8. A one-dimensional example

In this section, we briefly discuss a one-dimensional example, to clarify some of the notions of e_n-minimality and e_n-stability. We consider

$$F(p,z) := p^2 - z^2, \qquad \Omega := \mathbb{R}$$

$$\text{and } u(s) := \begin{cases} \cos(s+\pi/2), & \text{if } s \leqslant -\pi/2, \\ 1, & \text{if } -\pi/2 < s < \pi/2, \\ \cos(s-\pi/2), & \text{if } s \geqslant \pi/2. \end{cases} \tag{8.1}$$

Proposition 8.1. *The function u in* (8.1) *is e_1-stable in $(-\pi,\pi)$, (see Definition 7.3).*

Proof. Notice that $u \in C^{1,1}(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{-\pi/2, \pi/2\})$. We prove that

$$\begin{aligned} &\text{for any } R \in (0,\pi), \text{ any } \delta \in (0,\pi/2) \\ &\text{and any Lipschitz function } \varphi \text{ supported in } (-R,R) \\ &\text{with } \varphi \leqslant 0 \text{ in } [-\pi/2,\pi/2] \text{ and } \varphi = 0 \text{ in } [\delta-(\pi/2),(\pi/2)-\delta] \\ &\text{we have that } \mathscr{E}_R(u+\varphi) \geqslant \mathscr{E}_R(u). \end{aligned} \tag{8.2}$$

To prove it, we may suppose $R \in (\pi/2, \pi)$, and we define $I := (-R, R) \setminus [-\pi/2, \pi/2]$, $J_- := (-R,\, \delta - (\pi/2))$, $J_+ := ((\pi/2) - \delta,\, R)$ and $J := J_- \cup J_+$. Given $\ell > 0$, we also denote by $\lambda_\ell = \pi^2/\ell^2$ the first Dirichlet eigenvalue in the interval of length ℓ. By taking $\ell := R - (\pi/2) + \delta \in (0, \pi)$, we obtain that

$$\int_{J_\pm} \dot\varphi^2\, ds \geqslant \lambda_\ell \int_{J_\pm} \varphi^2\, ds \geqslant \int_{J_\pm} \varphi^2\, ds$$

therefore

$$\int_J \varphi^2 - \dot\varphi^2\, ds \leqslant 0.$$

So, we compute:

$$\begin{aligned}
&\mathscr{E}_R(u) - \mathscr{E}_R(u+\varphi)\\
&\qquad = 2\int_{-R}^{R} u\varphi - \dot u\dot\varphi\, ds + \int_{-R}^{R} \varphi^2 - \dot\varphi^2\, ds\\
&\qquad = 2\int_{-\pi/2}^{\pi/2} \varphi\, ds + 2\int_I u\varphi - \dot u\dot\varphi\, ds + \int_J \varphi^2 - \dot\varphi^2\, ds\\
&\qquad \leqslant 0 - 2\int_I \ddot u\varphi + \dot u\dot\varphi\, ds + 0\\
&\qquad = -2\int_I \frac{d}{ds}(\dot u\varphi)\, ds\\
&\qquad = (\dot u\varphi)(-\pi/2) - (\dot u\varphi)(\pi/2)\\
&\qquad = 0,
\end{aligned}$$

which establishes (8.2).

Now let $v \in D^t_{R,1}$, with $0 < t < \delta$. Then we define $\varphi(s) := v(s) - u(s)$. Notice that φ is Lipschitz, with $\|\varphi\|_{C^{0,1}(\mathbb{R})} \leqslant C\|u\|_{C^{1,1}(\mathbb{R})} t$, and supported inside $(-R, R)$. Also, $v \leqslant 1$, since v is a deformation of u and $u \leqslant 1$. Therefore, for any $s \in [-\pi/2, \pi/2]$, we see that $\varphi(s) = v(s) - 1 \leqslant 0$. Finally, since v is a horizontal deformation of u of size t, we have that

$$\inf_{[\delta-(\pi/2),(\pi/2)-\delta]} v \geqslant \inf_{[\delta-(\pi/2)-t,(\pi/2)-\delta+t]} u \geqslant \inf_{[-(\pi/2),(\pi/2)]} u = 1.$$

Consequently, if $s \in [\delta - (\pi/2), (\pi/2) - \delta]$ we have that $v(s) = 1$ and $\varphi(s) = 0$. So we can apply (8.2) and obtain $\mathscr{E}_R(v) = \mathscr{E}_R(u + \varphi) \geqslant \mathscr{E}_R(u)$. □

As a consequence of Proposition 8.1, we have that e_n-minimizers are not necessarily critical for the energy $\mathscr{E}$ at the points where the gradient vanishes. We recall that the situation for $\{e_n, e_{n+1}\}$-stable solutions was different, since in that case the criticality of the energy functional was granted by the vertical perturbations.

The example in (8.1) may be modified in order to obtain $\{\delta, e_1\}$-minimality in the whole of $\mathbb{R}$. For instance one may consider:

$$F(p,z) := p^2 - \max\{z,0\}^2, \qquad \Omega := \mathbb{R}$$

$$\text{and } u(s) := \begin{cases} s+\pi, & \text{if } s < -\pi \\ \cos(s+\pi/2), & \text{if } s \in [-\pi,-\pi/2], \\ 1, & \text{if } -\pi/2 < s < \pi/2, \\ \cos(s-\pi/2), & \text{if } s \in [\pi/2,\pi] \\ \pi - s, & \text{if } s > \pi. \end{cases} \tag{8.3}$$

Then the proof of Proposition 8.1 may be easily modified to obtain:

Proposition 8.2. *The function u in* (8.3) *is a $\{\delta, e_1\}$-minimizer, for any $\delta \in (0,\pi/2)$, according to Definition 7.6.*

This shows that the statement of Theorem 7.7 is optimal, since (8.3) provides an example of $\{\delta, e_1\}$-minimizer which is monotone on intervals of length $2\delta < \pi$ but not on intervals of larger length.

9. Proofs of some remarks

Proof of Remark 2.6. We consider a continuous function u which is monotone on each line in $\mathbb{R}^n$. We show that

$$\text{for any } t \in \mathbb{R}, \text{ the sublevel } \{u < t\} \text{ is a half-space} \tag{9.1}$$

(unless it is empty). From this, it follows that, for different values of t, $\partial\{u < t\}$ gives a collection of hyperplanes (which are parallel, since the level sets $\{u = t\}$ cannot intersect for different values of t), and so u is one-dimensional.

To prove (9.1), first we remark that, from the monotonicity on each line of u, it follows that

$$\text{both } \{u < t\} \text{ and } \{u \geqslant t\} \text{ are convex sets.} \tag{9.2}$$

Then, we take $p \in \{u < t\}$. Since u is continuous, there exists $\varrho > 0$ such that

$$B_\varrho(p) \subseteq \{u < t\}. \tag{9.3}$$

We enlarge ϱ till there exists a point

$$q \in \{u = t\} \cap \partial B_\varrho(p). \tag{9.4}$$

We denote by Π_- the open halfspace tangent to $B_\varrho(p)$ at q that contains p, and by Π_+ the closed halfspace tangent to $B_\varrho(p)$ at q that does not contain p.

By looking at all the lines passing through q, we deduce from (9.2), (9.3) and (9.4) that

$$\Pi_- \subseteq \{u < t\} \tag{9.5}$$

and

$$\Pi_+ \subseteq \{u \geqslant t\}. \tag{9.6}$$

By taking the complementary sets in (9.6) and noticing that Π_+ is the complement of Π_-, we conclude that

$$\Pi_- \supseteq \{u < t\}.$$

This and (9.5) give that $\Pi_- = \{u < t\}$, proving (9.1). □

Proof of Remark 2.9. Suppose that $u \in C^2(\mathbb{R}^n)$ is a classical stable solution, i.e. a critical point of the energy functional satisfying (2.6). Since $F \in C^2$, we have that for any Lipschitz function φ supported in a given ball B_R,

$$\begin{aligned} &F(\nabla u + t\nabla\varphi, u + t\varphi, x) - F(\nabla u, u, x) \\ &= t\Big(F_{p_i}\varphi_i + F_z\varphi\Big) + \frac{t^2}{2}\Big(F_{p_ip_j}\varphi_i\varphi_j + F_{zz}\varphi^2 + 2F_{p_iz}\varphi\varphi_i\Big) + o(t^2), \end{aligned}$$

where the derivatives of F are evaluated at $(\nabla u, u, x)$. Notice that $o(t^2)$ above only depends on the Lipschitz norms of φ and u in B_R, and on the C^2-norm of F in a bounded set (depending on R as well). When we integrate the equality above over B_R, the term of order t disappears since u is a critical point, therefore we obtain

$$\begin{aligned} &\mathscr{E}_R(u + t\varphi) - \mathscr{E}_R(u) \\ &\quad = \frac{t^2}{2}\int_{B_R} \Big(F_{p_ip_j}(\zeta)\varphi_i\varphi_j + F_{zz}(\zeta)\varphi^2 + 2F_{p_iz}(\zeta)\varphi\varphi_i\Big)\,dx + o(t^2). \end{aligned} \tag{9.7}$$

Dividing by t^2 and recalling (2.6), we conclude that

$$\int_{B_R} \Big(F_{p_ip_j}(\zeta)\varphi_i\varphi_j + F_{zz}(\zeta)\varphi^2 + 2F_{p_iz}(\zeta)\varphi\varphi_i\Big)\,dx \geqslant 0.$$

Hence, going back to (9.7), we obtain that

$$\mathscr{E}_R(u + t\varphi) - \mathscr{E}_R(u) \geqslant o(t^2). \tag{9.8}$$

Now, given $w \in \mathscr{D}^t_R(u)$, we take $\varphi := (w - u)/t$. Notice that the Lipschitz norm of φ is bounded uniformly in t, therefore (9.8) implies (2.5) and so u is $\{e_n, e_{n+1}\}$-stable.

Viceversa, suppose that u is $\{e_n, e_{n+1}\}$-stable. Then u is a critical point and (2.5) implies (2.6) by choosing $w := u + t\varphi$ and taking ϵ arbitrarily small. This shows that u is a stable solution. □

Proof of Remark 2.13. We define $e_0(s) := s$ and then recursively

$$e_k(s) := \exp(e_{k-1}(s)) = \underbrace{\exp\circ\cdots\circ\exp}_{k \text{ times}} s$$

for any $k \in \mathbb{N}$, $k \geqslant 1$. Let $\theta_k(R) := e_k(\sqrt{\ell_k(R)})$. Notice that

$$\ell_{k+1}(\theta_k(R)) = \log \sqrt{\ell_k(R)} = \frac{1}{2} \log(\ell_k(R)) = \frac{\ell_{k+1}(R)}{2}. \tag{9.9}$$

By induction over k, one sees that

$$\ell_k'(r) = \big(\pi_{k-1}(r)\big)^{-1}, \tag{9.10}$$

where the notation in (2.8) was used together with the setting $\pi_{-1}(r) := 1$ (in this way, $\pi_k(r) = \ell_k(r)\pi_{k-1}(r)$ for any $k \in \mathbb{N}$). We obtain that

$$\begin{aligned} \pi_k'(r) &= \sum_{m=0}^{k} \prod_{\substack{0\leqslant j\leqslant k \\ j\neq m}} \ell_j(r)\ell_m'(r) \\ &= \sum_{m=0}^{k} \prod_{j=m+1}^{k} \ell_j(r) \\ &\leqslant (k+1) \prod_{j=1}^{k} \ell_j(r) \\ &= (k+1) r^{-1} \pi_k(r) \end{aligned}$$

for large r, and so

$$-\frac{d}{dr}\big(\pi_k(r)\big)^{-2} = 2\big(\pi_k(r)\big)^{-3}\pi_k'(r) \leqslant 2(k+1)r^{-1}\big(\pi_k(r)\big)^{-2} \tag{9.11}$$

for large r. Now, recalling (9.9), we modify (5.1) as follows:

$$\psi_R(s) := \begin{cases} 1, & \text{if } 0 \leqslant s \leqslant \theta_k(R), \\ 2 - \dfrac{2\ell_{k+1}(s)}{\ell_{k+1}(R)}, & \text{if } \theta_k(R) < s \leqslant R, \\ 0, & \text{if } s > R. \end{cases} \tag{9.12}$$

From (9.12) and (9.10) we see that

$$\psi_R'(s) = \begin{cases} -\dfrac{2}{\ell_{k+1}(R)\pi_k(s)}, & \text{if } \theta_k(R) < s \leqslant R, \\ 0, & \text{otherwise.} \end{cases} \tag{9.13}$$

Notice that (9.12) and (9.13) reduce to (5.1) and (5.2) respectively when $k = 0$. Then, we can argue as in Section 5. In this case, (5.4) gets replaced

by

$$
\begin{aligned}
&\mathscr{E}_R(u^+_{R,t}) + \mathscr{E}_R(u^-_{R,t}) - 2\mathscr{E}_R(u) \\
&\qquad \leqslant C\frac{t^2}{\big(\ell_{k+1}(R)\big)^2}\int_{\Omega\cap(B_R\setminus B_{\theta_k(R)})}\frac{|F_{pp}(\nabla u,u,x')|\,|\nabla u|^2}{\big(\pi_k(|x|)\big)^2}\,dx \\
&\qquad = C\frac{t^2}{\big(\ell_{k+1}(R)\big)^2}\int_{\Omega\cap(B_R\setminus B_{\theta_k(R)})}|F_{pp}(\nabla u,u,x')|\,|\nabla u|^2\,\sigma(|x|)\,dx,
\end{aligned}
\tag{9.14}
$$

where

$$\sigma(r) := \big(\pi_k(r)\big)^{-2}.$$

Therefore, we recall (5.5) and we notice that, in this case, $a(r) \leqslant Cr\pi_k(r)$ for large r, thanks to (2.9). So we use (9.11) and (9.10), and, instead of (5.6), in this case we bound the last integral on right hand side of (9.14) in polar coordinates by

$$
\begin{aligned}
\int_{\theta_n(R)}^R a'(r)\sigma(r)\,dr &\leqslant a(R)\sigma(R) - \int_{\theta_n(R)}^R a(r)\sigma'(r)\,dr \\
&\leqslant CR\big(\pi_k(R)\big)^{-1} + C\int_{\theta_n(R)}^R \big(\pi_k(r)\big)^{-1}\,dr \\
&= CR\big(\pi_k(R)\big)^{-1} + C\int_{\theta_n(R)}^R \ell'_{k+1}(r)\,dr \\
&\leqslant C + C\ell_{k+1}(R).
\end{aligned}
$$

Therefore, (9.14) gives in this case

$$
\begin{aligned}
&\limsup_{R\to+\infty}\sup_{t\in(0,\theta_k(R)/4)} t^{-2}\Big(\mathscr{E}_R(u^+_{R,t}) + \mathscr{E}_R(u^-_{R,t}) - 2\mathscr{E}_R(u)\Big) \\
&\qquad \leqslant \limsup_{R\to+\infty}\sup_{t\in(0,\theta_k(R)/4)} \frac{C}{\big(\ell_{k+1}(R)\big)^2}\big(1+\ell_{k+1}(R)\big) = 0,
\end{aligned}
$$

which replaces (5.7) in this case. □

Proof of Remark 2.14. Here we construct a one-dimensional example of a Lipschitz function $u : \mathbb{R} \to \mathbb{R}$ that satisfies (2.4) and that minimizes the energy with respect to any e_n-Lipschitz deformation, without being monotone. For this we take $u(t) := |t|$, $\Omega := \mathbb{R}$ and $F := |p|^2$. Then, (2.4) is obvious, and clearly u is not monotone. Let us check that it is minimal with respect to any e_n-Lipschitz deformation, as described in Definition 2.1: for this let ψ be Lipschitz and supported in $(-R, R)$, with $|\psi'| < 1$, and $v(t) = u(t+\psi(t)) = |t+\psi(t)|$. We have

$$|v'(t)|^2 - |u'(t)|^2 = (1+\psi'(t))^2 - 1 = 2\psi'(t) + (\psi'(t))^2$$

for almost any $t \in (-R, R)$. Therefore, if we integrate over $(-R, R)$ and we use that $\psi(-R) = 0 = \psi(R)$, we obtain

$$\mathscr{E}_R(v) - \mathscr{E}_R(u) = \int_{-R}^{R} (\psi'(t))^2 \, dt \geqslant 0,$$

which is the minimality with respect to e_n-Lipschitz deformations.

It is worth noticing that u is not an e_n-minimizer, since piecewise e_n-Lipschitz deformations may decrease the energy (this justifies the importance of Definition 2.2). To show this, we take $R := 2$,

$$\psi^{(1)}(t) := \begin{cases} -\dfrac{2+t}{3}, & \text{if } t \in [-2, 1], \\ \\ t-2, & \text{if } t \in (1, 2], \end{cases}$$

$$\text{and} \qquad \psi^{(2)}(t) := \begin{cases} t+2, & \text{if } t \in [-2, -1], \\ \\ \dfrac{2-t}{3}, & \text{if } t \in (-1, 2]. \end{cases}$$

Let also $v^{(i)}(t) := u(t + \psi^{(i)}(t))$ and

$$v(t) := \begin{cases} v^{(1)}(t), & \text{if } t \in [-2, 0], \\ v^{(2)}(t), & \text{if } t \in (0, 2]. \end{cases}$$

Then v is a piecewise e_n-Lipschitz deformation of u according to Definition 2.2 and one may explicitly compute that

$$v(t) = \frac{2(|t| + 1)}{3}.$$

In particular, $\mathscr{E}_R(v) = (8/9)R < 2R = \mathscr{E}_R(u)$, which shows that u is not an e_n-minimizer. □

Proof of Remark 3.2. We define $a(t)$, $\lambda_i(t)$, $\Lambda_i(t)$ and $A_{ij}(p)$ as in [19] (see, in particular, formulas (1.4)–(1.6) there). To avoid confusion with the notation here, the function F introduced below (1.6) in [19] will be denoted by $\tilde{F}$. The goal is to apply Theorem 2.10 with $F(p, z, x) := \Lambda_2(|p|) + \tilde{F}(z)$ (since this and Remark 2.6 here plainly imply Theorems 1.1 and 1.2 in [19]). For this, we need to check the convexity of F in p and conditions (2.2) and (2.4). We may focus on the case $n = 3$, i.e. on the case of Theorem 1.2 of [19] (this allows us to take also assumptions (B1) and (B2) in [19]). From (1.6) and (1.5) in [19], we see that

$$F_{p_i}(p, z, x) = \lambda_2(|p|) \, p_i = a(|p|) \, p_i$$

and so

$$F_{p_i p_j}(p, z, x) = a(|p|) \, \delta_{ij} + a'(|p|) \, |p|^{-1} p_i p_j = A_{ij}(p).$$

Therefore, Lemma 2.1 in [19] gives the desired convexity of F and it implies that

(9.15)

$$|F_{pp}(p, z, x)| \text{ is bounded from above and below by } C\big(\lambda_1(|p|) + \lambda_2(|p|)\big).$$

On the other hand, by Lemma 4.2 of [19], we have that, if $|p| \leqslant M$, then

$$\lambda_1(|p|) \leqslant C_M \lambda_2(|p|) \tag{9.16}$$

and

$$\lambda_2(|p+q|) \leqslant C_M \lambda_2(|p|) \tag{9.17}$$

if $|q| \leqslant |p|/2$, for suitable $C_M > 0$ (possibly varying line after line). By plugging (9.16) into (9.15) we obtain that, if $|p| \leqslant M$,

$$|F_{pp}(p,z,x)| \leqslant C_M \lambda_2(|p|). \tag{9.18}$$

Using (9.18) and (9.17) we see that if $2|q| \leqslant |p| \leqslant M$,

$$|F_{pp}(p+q,z,x)| \leqslant C_M \lambda_2(|p+q|) \leqslant C_M \lambda_2(|p|) \leqslant C_M |F_{pp}(p,z,x)|,$$

which gives (2.2) (notice that we may suppose $|p| = |\nabla u| \leqslant M$ in this case).

Moreover, using (9.18) here and (4.3) of [19], we obtain

$$|F_{pp}(p,z,x)|\,|p|^2 \leqslant C_M \lambda_2(|p|)\,|p|^2 = C_M a(|p|)\,|p|^2 \leqslant \Lambda_2(|p|).$$

This and (5.16) in [19] imply

$$\int_{B_R} |F_{pp}(\nabla u, u, x)|\,dx \leqslant C_M \int_{B_R} \Lambda_2(|\nabla u|)\,dx \leqslant C_M R^2.$$

This shows that (2.4) holds true in this case: so we may use Theorem 2.10, then recall Remark 2.6, and obtain the one-dimensional results of Theorems 1.1 and 1.2 of [19]. □

References

[1] G. Alberti, L. Ambrosio, X. Cabré, *On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property*, Acta Appl. Math. 65, no. 1-3, 9–33 (2001).

[2] H.W. Alt, L. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math 325, 105–144 (1981).

[3] H. W. Alt, L. A. Caffarelli, A. Friedman, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc. 282, 431–461 (1984).

[4] L. Ambrosio, X. Cabré, *Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi*, J. Amer. Math. Soc. 13, no. 4, 725–739 (2000).

[5] B. Barrios Barrera, A. Figalli, E. Valdinoci, *Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), DOI: 10.2422/2036-2145.201202_007, `arxiv.org/abs/1202.4606`

[6] H. Berestycki, L. Caffarelli, L. Nirenberg, *Further qualitative properties for elliptic equations in unbounded domains*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25, no. 1-2, 69–94 (1997-98).

[7] X. Cabré, E. Cinti, *Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian*, Discrete Contin. Dyn. Syst. 28, no. 3, 1179–1206 (2010).

[8] X. Cabré, E. Cinti, *Fractional diffusion equations: energy estimates and 1-D symmetry in dimension* 3, preprint.

[9] L. Caffarelli, D. Jerison, C.E. Kenig, *Global energy minimizers for free boundary problems and full regularity in three dimension*, Contemp. Math., 350, Amer. Math. Soc., Providence, RI, 83–97 (2004).

[10] X. Cabré, J. Solà-Morales, *Layer solutions in a half-space for boundary reactions*, Comm. Pure Appl. Math. 58, no. 12, 1678–1732 (2005).

[11] X. Cabré, Y. Sire, *Nonlinear equations for fractional Laplacians II: existence, uniqueness, and qualitative properties of solutions*, preprint, arxiv.org/abs/1111.0796

[12] L. Caffarelli, J.-M. Roquejoffre, O. Savin, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. 63, no. 9, 1111–1144 (2010).

[13] L. Caffarelli, L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations 32, no. 7-9, 1245–1260 (2007).

[14] L. Caffarelli, E. Valdinoci, *Regularity properties of nonlocal minimal surfaces via limiting arguments*, preprint, arxiv.org/abs/1105.1158

[15] E. De Giorgi, *Convergence problems for functionals and operators*, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis, Bologna, Pitagora (1979).

[16] D. De Silva, O. Savin, *Regularity of Lipschitz free boundaries for the thin one-phase problem*, preprint, arxiv.org/abs/1205.1755

[17] A. Farina, *Propriétés qualitatives de solutions d'équations et systèmes d'équations non-linéaires*, Habilitation à diriger des recherches, Paris VI (2002).

[18] A. Farina, *One-dimensional symmetry for solutions of quasilinear equations in* $\mathbb{R}^2$, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 6, no. 3, 685–692 (2003).

[19] A. Farina, B. Sciunzi, E. Valdinoci, *Bernstein and De Giorgi type problems: new results via a geometric approach*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 7, no. 4, 741–791 (2008).

[20] N. Ghoussoub, C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann. 311, no. 3, 481–491 (1998).

[21] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001, xiv+517 pp.

[22] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, Birkhäuser-Verlag, Basel, 1984, xii+240 pp.

[23] L. Moschini, *New Liouville theorems for linear second order degenerate elliptic equations in divergence form*, Ann. I. H. Poincaré 22, no. 1, 11–23 (2005).

[24] O. Savin, *Regularity of flat level sets in phase transitions*, Ann. of Math. 169, no. 1, 41–78 (2009).

[25] O. Savin, E. Valdinoci, *Regularity of nonlocal minimal cones in dimension 2*, Calc. Var. Partial Differential Equations, DOI: 10.1007/s00526-012-0539-7, http://www.springerlink.com/content/467n313161531332/fulltext.pdf